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THEORY OF THE BOLTZMANN EQUATION

Harold Grad

August 1964

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August 1964

Preprint of paper presented at
Eleventh International Congress of Applied Mechanics
Munich, August 1964

Air Force Office of Scientific Research
Grant No. AF-AFOSR-62-266

Abstract

We present a survey of recent developments in the theory of the Boltzmann equation for a dilute monatomic gas. These advances are characterized by increased sophistication in mathematical techniques, improved understanding of the significance of older approximate results, and a general trend away from *ad hoc* and toward more precise mathematical procedures. In the singular limit of small mean free path, the traditional Hilbert and Chapman-Enskog expansions have been shown to be asymptotic to true solutions of the Boltzmann equation, but only when the variables are appropriately interpreted. At the opposite extreme of large mean free path where the behavior is again nonuniform, the precise mathematical singularities have been exposed. Over the whole range of linear problems the presence of a continuous spectrum in both the collision operator and the streaming operator points to the inadequacy of traditional exponential "normal mode" expansions in both initial and boundary value problems. Results in all the regimes have been tied together by the overall qualitative understanding given by a more comprehensive existence theory which, for the first time, is broad enough to encompass the transition to macroscopic continuum flow.

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1. Introduction

The purpose of this article is to bring up to date earlier surveys[1,2] of the state of the art in the theory of the Boltzmann equation. Historically, the principal methods of attack have been connected with normal solutions and transport coefficients (Hilbert and Chapman-Enskog), polynomial and other more-or-less arbitrary approximations to the distribution function, integral iterations, and methods specific to models of the Boltzmann equation (e.g. Fourier transform). Each method has seen decisive advances in recent years. Another attack, of very recent appearance, is direct numerical computation; this method is still largely exploratory.

The significance of the Hilbert and Chapman-Enskog theories as formal asymptotic expansions of solutions of the Boltzmann equation was described qualitatively in 1958[3] and in full mathematical detail more recently[4,5] (see Sect. 3). The crucial qualitative point is the necessity to reinterpret

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1. Harold Grad, "Principles of the Kinetic Theory of Gases" in Handbuch der Physik, ed. S. Flugge, Vol.XII (Springer Verlag, 1958).
2. Harold Grad, "Theory of Rarefied Gases" in Rarefied Gas Dynamics, ed. F. Devienne (Pergamon Press, 1960).
3. Reference 1, ¶ 26.
4. Harold Grad, Phys. Fluids 6, 147 (1963).
5. Harold Grad, "Asymptotic Equivalence of the Navier-Stokes and Nonlinear Boltzmann Equations", to appear in Proc. of American Mathematical Society Symposium on Applications of Partial Differential Equations, New York, April 1964.

the variables that arise naturally in the Hilbert and Chapman-Enskog expansions as suitably modified macroscopic properties of the gas, not simply the fluid state. As a result of this generalized interpretation, the Chapman-Enskog expansion has been found to be representative of very general solutions of the Boltzmann equation in the limit of small mean free path.

Polynomial and moment approximations, although to some extent forseen by Maxwell, overcame an almost exclusive preoccupation with transport coefficients only in 1949[6] and were later extensively developed for the linear equation with Maxwellian potential by Uhlenbeck and Wang Chang[7] and by Mott-Smith[8] (also more recently by Sirovich[9]). More complicated and more accurate (but essentially arbitrary) approximations followed, principally the half-range[10,11] and the bimodal.[12,13]

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6. Harold Grad, Communs. Pure and Appl. Math. 2, 331 (1949).

7. G. E. Uhlenbeck and C. S. Wang Chang, "On the propagation of sound in monatomic gases", Univ. of Michigan, Engineering Research Inst. (1952).

8. H. M. Mott-Smith, "A New Approach in the Kinetic Theory of Gases", MIT, Lincoln Laboratory (1954).

9. L. Sirovich, Phys. Fluids 6, 218 (1963).

10. E. P. Gross, E. A. Jackson, and S. Ziering, Ann. Phys. 1, 141 (1957).

11. Lester Lees, "A kinetic theory description of plane compressible Couette flow", GALCIT Hypersonic Research Project, Memo No. 51 (1959).

12. H. M. Mott-Smith, Phys. Rev. 82, 885 (1951).

13. H. Salwen, C. E. Grosch, and S. Ziering, Phys. Fluids 7, 180 (1964).

But only recently have these methods been correctly recognized as being ad hoc approximations which ignore the analytical peculiarities of exact solutions. Although a judiciously selected low order approximation may give a good overall description, greater accuracy is not easily forthcoming in higher order, and even qualitative features may be lost if the precise analytical peculiarities (notably nonuniform behavior and the continuous spectrum) are not taken into account[14,15] (see Sect. 5). In certain nonlinear flows the question of convergence of polynomial approximations can even be given a definite negative answer in certain circumstances.[16]

Integral iterations, although formulated by Enskog[17] and used for abstract existence theory[18,19,20,21], have only recently received substantial use in problems in which

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14. C. Cercignani, Ann. Phys. 20. 219 (1962).

15. Harold Weitzner, "Steady State Oscillations in a Gas" to appear in Proceedings of the Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

16. L. W. Holway, Jr., Phys. Fluids 7, 911 (1964).

17. D. Enskog, Ark. Mat., Astronom. Phys. Ser. A 21, 1 (1928).

18. T. Carleman, Acta math., Stockh. 60, 91 (1933).

19. E. Wild, Proc. Cambridge Phil. Soc. 47, 602 (1951).

20. D. Morgenstern, Proc. Nat. Acad. Sci. 40, 719 (1954).

21. Reference 1, Chap. III.

the method is accurate, viz., in near free flow[22], flow near a boundary[23], and in problems where a good approximation is known as a first iterate.[24] The first iterate is even surprisingly accurate in problems of near free flow (cf. Sect. 4).

As to models of the Boltzmann equation[25,26] a number of essentially exact results have been obtained by use of Fourier and Laplace transform, e.g., the initial value problem[27], flow around a small object[28,29], and flow in a half space.[14,15]

Most of the advances listed above are tied together with the thread of mathematically singular and nonuniform

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22. D. R. Willis, "The Effect of the Molecular Model on Solutions to Linearized Couette Flow with Large Knudsen Number" in Rarefied Gas Dynamics, ed. L. Talbot (Academic Press, 1961).

23. S. Ziering, L. Chi, and R. Fante, "Kinetic Theory of the Leading Edge", to appear in Proceedings of Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

24. H. W. Liepmann, R. Narasimha, M. T. Chahine, Phys. Fluids 5, 1313 (1962).

25. P. W. Welander, Arkiv Fysik 7, 507 (1954).

26. P. L. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. 94, 511 (1954).

27. L. Sirovich and J. K. Thurber, "Sound Propagation According to the Kinetic Theory of Gases" in Rarefied Gas Dynamics, ed. J. A. Laurmann (Academic Press, 1963).

28. Harold Grad, "Flow Equations in a Rarefied Atomsphere" in Proc. of Conference on Aerodynamics of the Upper Atmosphere, Rand Corp. (1959).

29. Marian H. Rose, "Drag on an Object in Nearly Free Molecular Flow", to appear in Phys. Fluids, August 1964.

behavior. Although the physical consequences may not always be great, the mathematical sensitivity which is thus revealed must not be disregarded in any attempt to derive accurate mathematical consequences from the equations.

2. General Theory

The development of a comprehensive theory of existence and qualitative behavior of solutions of the Boltzmann equation serves several "practical" purposes. The first is to help one recognize whether the solution of a specific problem is representative of more general cases or whether it may be exceptional. For example, we recall that almost all the extrapolations that were drawn from a very ingenious explicit solution of an exact nonlinear Boltzmann equation[30] have been found to be nonrepresentative of the behavior of the typical solution.[4]

A more basic goal of a general theory is to determine whether or not there exist any solutions at all. Such a failure is always a conceptual possibility no matter how convincing are the physical arguments which suggest the

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30. E. Ikenberry and C. Truesdell, J. Ratl. Mech. Anal. 5, 1 (1956).

validity of the equation and no matter how plausible are the results derived by nonrigorous approximate means. The reason is the inherent sensitivity of mathematical structures to arbitrarily small perturbations which are undectable physically. In other words, the validity of an equation such as Boltzmann's must be based on studies of internal mathematical consistency as well as on external confrontation with experiment.

With regard to the Boltzmann equation, there is at the present time only the beginning of a theory pointing to the existence of nonlinear solutions in the large.[5] On the other hand, despite pessimistic statements in the literature,[31,32] there is no specific evidence to the contrary other than the fact that it is too much to expect nonlinear equations in general to have solutions in the large (existence of solutions would have to result from specific peculiarities of the Boltzmann equation). There are powerful existence theorems for spatially homogeneous problems in which fluid dynamics plays no role. But the question of the existence of spatially dependent solutions is certainly as difficult as the corresponding question in nonlinear fluid dynamics, and this connection is, in fact, the most important

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31. D. Morgenstern, J. Ratl. Mech. Anal. 4, 533 (1955).

32. Ya. Povzner, Math. Sbornik, 62 (1962).

heuristic argument for a belief in the reasonable behavior of solutions of the Boltzmann equation in the large.

Turning to another difficulty, there is no evidence at this time that even the linear equation has any validity for potentials of infinite total cross-section. We would expect no physical differences between an infinite range but rapidly decaying intermolecular potential and a closely approximating cutoff potential; but the mathematics seems to be extraordinarily sensitive to this point (we must distinguish solution of the Boltzmann equation from solution of the integral equation which arises in transport theory and which is known to be insensitive to "small" changes in the potential).

The first results on existance of solutions to the Boltzmann equation were obtained by Carleman for hard sphere molecules.[18] He obtained a strong nonlinear result in the large but only in a spatially homogeneous problem. Although the mathematics is quite difficult, this result is physically trivial and, in particular, includes no fluid dynamics. Similar results (spatially homogeneous) were proved much more easily for the mathematically simpler cutoff Maxwellian potential.[19,20]

Carleman also gave an existence proof for the linear hard sphere equation including spatial variation[33], but with

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33. T. Carleman, Problèmes Mathématiques dans la Théorie Cinétique des Gaz (Almqvist and Wiksells, Uppsala, 1957).

a growth estimate $e^{t/\tau}$ (τ is the mean free time). This is a catastrophically poor estimate on any macroscopic time scale. Recently, the theory of the linear equation has been generalized to include essentially arbitrary cutoff repulsive potentials, and it has been extended to show not only existence but smoothness and general good behavior of solutions, in particular boundedness for all time, uniformly in the mean free path.[34] Moreover, all solutions are shown to approach equilibrium (under suitable boundary conditions).[35] This is a nontrivial extension of Boltzmann's H-theorem which, in its original formulation, can show only an approach to a local Maxwellian in a spatially varying problem. It is the fact that the solution does not approach too closely to the local Maxwellian which provides a residual irreversibility and allows an eventual fluid-like decay to equilibrium.

Based on this strong linear theory, a limited nonlinear theory, for excursions from linearity which are not too large, has been obtained for times of interest on a macroscopic fluid dynamic time scale.[5] Thus the development of a complete nonlinear theory would seem to be subject to technical rather than basic difficulties. It is noteworthy that solutions of

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34. Harold Grad, "Asymptotic Theory of the Boltzmann Equation, II" in Rarefied Gas Dynamics, ed. J. A. Laurmann, Vol. I (Academic Press, 1963).

35. Harold Grad, "On Boltzmann's H-theorem", to be published in Proceedings of SIAM von Karman Symposium, Washington, May 1964.

the linear Boltzmann equation approximate the true nonlinear solutions better than might be expected on the basis of plausible estimates.[5]

For certain altered versions of the Boltzmann equation chosen for mathematical convenience, strong nonlinear existence theorems in the large have been proves.[31,32] But these equations contain arbitrary averaging or smoothing operators, and the relation to an actual gas is not established. These replacements for the Boltzmann equation should not be confused with so-called "model" equations[25,26] which do retain a heuristic physical meaning but do not appreciably simplify nonlinear existence theory.

3. Small Mean Free Path

On the basis of physical evidence we expect the detailed description of a gas by means of a molecular distribution function to be adequately approximated by an apparently cruder macroscopic fluid description when the mean free path is small. Mathematically this must correspond to a very singular limit. The appropriate mathematical formulation, introduced by Hilbert[36], is to place a large

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36. D. Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen (Wien, J. B. Teubner, 1924) (also Chelsea Publishing Co., New York, 1953).

factor $1/\varepsilon$ (ε is a scale factor for the mean free path or mean collision time) in front of the quadratic collision term in the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = \frac{1}{\varepsilon} Q(f,f) \ . \tag{3.1}$$

Despite the fact that $\varepsilon = 0$ is singular, Hilbert looks for solutions in the form

$$f = f^{o} + \varepsilon f^{1} + \varepsilon^{2} f^{2} + \ldots \ . \tag{3.2}$$

Inserting the power series and equating like terms, we find that the successive terms f^{n} are determined uniquely by an algorithm (successively solving integral and partial differential equations) which requires specification of <u>fluid</u> initial data only. We can interpret this result as a statement that the expansion (3.2) is valid for a special class of solutions which has the property that the distribution function f at a given instant is uniquely determined by the instantaneous fluid state (as a functional, not just a point function since the f^{n} depend on derivatives of the fluid variables). This observation leads naturally to the Chapman-Enskog theory. If we evaluate the special Hilbert f in its dependence on the fluid state, we can compute from it the stress tensor and heat flow vector as moments of f; they will also be expressed as power series in ε involving successively higher derivatives of the fluid variables. The Boltzmann

equation can now be bypassed, and the time evolution of the fluid state can be obtained from the equations of conservation of mass, momentum, and energy which are self-contained in the fluid variables (to any given order of ε).

At this state both the Hilbert and Chapman-Enskog expansions are only formal; the exact relation to a hypothetical special class of solutions or possibly to general solutions of the Boltzmann equation remains to be discovered. Also, whereas the Hilbert algorithm can be carried out step by step by solving equations with an established theory, the more desirable Chapman-Enskog expansion requires the solution of nonstandard differential equations (at least beyond the Navier-Stokes level).

To fix the ideas let us now return to direct solution of the Boltzmann equation without recourse to power series and look at a problem with fixed molecular boundary conditions (specular, diffuse, or any other) and a fixed initial function $f(0)$ which is independent of ε. We suppose that the equation (3.1) has a solution $f(\xi,x,t;\ \varepsilon)$ and pose the question, what is the relation between the exact solution and the Hilbert or Chapman-Enskog expansions? We see immediately that the expansion (3.2) cannot solve this problem. For one, a Hilbert function (3.2) is explicitly and uniquely determined in its dependence on molecular velocity ξ whereas $f(0)$ should be arbitrary. Even if we attempt to restrict the

validity of (3.2) to special initial values f(0) of the Hilbert type, we are blocked by the fact that a Hilbert function (3.2) is necessarily a function of ε whereas f(0) has been chosen independent of ε. The resolution of this problem has been obtained in recent years by first proving the existence of the exact solution $f(\xi,x,t;\ \varepsilon)$ (for the linear[34] and slightly nonlinear[5] equations, see Sect. 2); and second by proving that with appropriate initial date the expansion (3.2) is <u>asymptotic</u> to the exact solution.[4] Specifically, the local Maxwellian f^o with fluid parameters taken from the solution of the inviscid Euler equations of fluid dynamics yields the result

$$\lim_{\varepsilon \to 0} f(\xi,x,t;\ \varepsilon) = f^o(\xi,x,t) \tag{3.3}$$

at any fixed positive time $t > 0$. Similarly, the Chapman-Enskog second approximation $f^o + \varepsilon f^1$ with fluid variation taken from the Navier-Stokes equations satisfies

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| f - (f^o + \varepsilon f^1) \right| = 0 \tag{3.4}$$

uniformly in $t > t_o > 0$ (t bounded away from zero) provided that the Navier-Stokes equations are given proper initial values -- <u>not</u> the actual initial fluid state as specified by f(0). The asymptotic series (3.2) takes over only after an initial transient (duration comparable to ε). The lasting effect of this transient on the fluid state is taken into

account by providing the Chapman-Enskog theory with an asymptotic initial fluid state which is derived from the entire initial state $f(0)$ in a complex way and is dependent on ε even though the actual initial fluid state is not.

This theory is mathematically complete for the linear and slightly nonlinear initial value problem; an algorithm has been given for the correct asymptotic fluid initial values for the general nonlinear problem.[4] By a plausible heuristic argument[3] one can show in the general nonlinear problem that the expansion (3.2) should be asymptotic to a true solution provided that one avoids not only the vicinity of $t = 0$, but also the boundary of the domain, and any shocks that may appear (initial, boundary, and shock layers). Because these expansions are only asymptotic, we are faced with the question of matching the expansions across the layers. Only the problem of the initial layer described above has been solved (initial slip). The boundary layer problem (boundary slip) is to find proper asymptotic boundary values for the Chapman-Enskog equations. Any special solution of the Boltzmann equation (e.g. heat flow between parallel plates) would yield as a byproduct the correct asymptotic boundary value for the special geometry. But only <u>ad hoc</u> approximations to solutions of boundary value problems have thus far been obtained (with the exception of a few solutions using a model of the Boltzmann equation[14,15]), see Sect. 5.

The problem of shock slip[3] is to find the correct jump conditions to replace the Hugoniot relations which are valid only to lowest order as appropriate to the inviscid Euler equations. It is a misleading fact that, whereas the Euler equations admit only a discontinuity across a shock, the Navier-Stokes equations yield a continuous transition. The Navier-Stokes solution, even though it is smooth, does not correctly approximate the Boltzmann solution uniformly across a shock wave. A treatment of the shock slip problem using the Navier-Stokes equations in the shock interior[37] is properly interpreted as only a rough approximation in general, but one which should be expected to give the correct result for weak shocks. The strictly correct usage of the Navier-Stokes equations as they relate to the limit of a Boltzmann solution for small mean free path is to solve using the Navier-Stokes equations on both sides of a discontinuous shock across which modified Hugoniot conditions (to order ε) are applied.

A numerical analysis of the classical Riemann problem of a ruptured diaphragm[38] shows qualitative agreement with this theory. After an initial definitely non-Maxwellian

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37. Ronald F. Probstein and Y. S. Pan, "Shock Structure and the Leading Edge Problem", in Rarefied Gas Dynamics, Vol. II, ed. J. A. Laurmann (Academic Press, 1963).

38. C. K. Chu, "A Kinetic-Theoretic Description of the Formation of a Shock Wave", to appear in Phys. Fluids.

transient, there emerge a moderately non-Maxwellian shock layer and a closely Maxwellian entropy layer. Quantitative agreement is not yet feasible.

4. Large Mean Free Path

This limiting case (large ε) also exhibits nonuniformities but not so radically as in the continuum limit (small ε) since the description of the state of the gas in the limit of free flow is by a distribution function just as for finite mean free path. The nonuniform behavior arises from two sources: first the universal presence of a residual class of particles which is dominated by collisions no matter how large is the mean free path; second, in the case of a molecule of infinite total cross-section only, the nonconvergence of the collision integral for a discontinuous distribution function of the type occurring in free flow.[39]

To be specific, consider a flow bounded by two parallel plates (e.g. heat flow or shear flow). A particle oriented in a grazing direction to a plate will travel many mean free paths between plates even if the normal spacing is much less

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39. We may note that although the infinite cross-section causes no difficulty in the formal evaluation of the Hilbert and Chapman-Enskog algorithms for small ε, the proofs seem to be very difficult.

than a mean free path. Thus the distribution function will differ from its free flow value by a perturbation of order $1/\varepsilon$ (ε is the Knudsen number and is large) except for a small region in velocity space. This nonuniform behavior is exhibited in an expansion of macroscopic variables (e.g. heat flow) which takes the form[22]

$$q_o + (\frac{1}{\varepsilon} \log \varepsilon) q_1 + \ldots \tag{4.1}$$

where q_o is the free flow value of the heat flow. In this problem the lowest order perturbation of the molecular distribution from free flow is correctly obtained uniformly in velocity by a <u>first integral iteration</u>. In other words, the correct distribution is obtained by a single iteration applied to the free flow solution. It is easy to verify that the singularity enters as $\varepsilon^{-2} \log \varepsilon$ in a cylindrical problem and $\varepsilon^{-3} \log \varepsilon$ in a domain of finite extent because the group of exceptional particles is of successively smaller dimension.

An entirely different type of near free flow problem is typified by flow around a small object (or flow through a small orifice). For definiteness let us take a body of finite size and allow the mean free path to grow. At any fixed distance from the body the flow will approach the classical free flow, separately given in the cone subtended by the body and its exterior. But this is evidently not the case one mean

free path or farther from the body; the free flow solution is not uniformly valid out to infinity.[28,40] To find the perturbation uniformly for all distances is a difficult problem. Further, the leading term in $1/\varepsilon$ for the distribution function is not correctly given by a single iteration as in the case of the parallel plates. However, it appears[41] that the perturbed distribution <u>at the body</u> is correctly given by one iteration (successive iterations would presumably improve the accuracy away from the body). Thus the drag, which requires knowledge of f only at the body, can presumably be correctly found by a single iteration. The entire perturbed distribution has been evaluated for a model of the Boltzmann equation and the drag computed.[29] The first integral iteration has also been computed, but for a different model[42], so the results cannot yet be compared.

There are two sources of nonuniformity in the expansion with respect to ε of the flow around an object. In addition to the effect of the slow particles as in the problem of parallel plates (or more precisely concentric spheres), there

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40. Y. Pao, "A Uniformly Valid Asymptotic Theory of Linearized Rarefied Gas Flows Under Nearly Free Molecular Conditions" to appear in Proceedings of Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

41. Conversation with D. R. Willis and Y. Pao.

42. G. J. Maslach, D. R. Willis, S. Tang, and D. Ko., "Recent Extension of Nearly Free Molecular Flow Experiment and Theory", to appear in Proceedings of Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

is also the effect of distant collisions on particles of all apeeds. Both are higher order effects which have not yet been computed.

The nonuniformity introduced by a power law potential of infinite total cross-section has been evaluated formally, but so far without a proof of convergence; the singularity in a parallel plate geometry turns out to be a fractional power of ε.[43]

5. Influence of the Spectrum

Essentially all the difficulty of the linear Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f + L(f) = 0 \tag{5.1}$$

arises from the interplay between the streaming term $\xi \cdot \nabla f$ and the collision term $L(f)$. They can be considered individually by studying free flow

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = 0 \tag{5.2}$$

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43. J. J. Smolderin, "Near Free Molecular Behavior of Gases with Infinite Collision Cross-section", to appear in Proceedings of Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

and the spatially homogeneous problem

$$\frac{\partial f}{\partial t} + L(f) = 0 \ . \tag{5.3}$$

The theory of Liouville's equation (5.2) is dominated by the fact that the spectrum of the streaming operator $\xi \cdot \nabla$ is continuous and comprises the entire imaginary axis. As a result, there is an approach to equilibrium for any initial distribution $f(0)$[44], but it is not universal, and the rate of decay to equilibrium may be exponential[45] or it may not and it will depend on the specific initial function $f(0)$.[44]

The collision operator L has also been found to have a continuum as part of its spectrum for almost all cutoff intermolecular potentials.[34] To be precise, L can be decomposed

$$L(f) = \nu f - K(f) \tag{5.4}$$

where $\nu(\xi)$ is the collision frequency of a molecule of speed ξ and K is nonsingular and compact. The continuous spectrum of L consists of the set of values taken by $\nu(\xi)$ as ξ ranges over all its values. For a hard potential $\nu \rightarrow \infty$ for large ξ; therefore, the continuum extends from some finite value to infinity. For a soft potential $\nu \rightarrow 0$ for large ξ, and the continuum extends from a finite value to the origin.

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44. Reference 1, ¶ 12; and Harold Grad, Communs. Pure and Appl. Math. XIV, 323 (1961), ¶ 5.

45. H. L. Frisch, Phys. Rev. 109, 22 (1958).

There may also be point eigenvalues and there is always a multiple eigenvalue at the origin corresponding to conservation of mass, momentum, and energy. Qualitatively, the relaxation to equilibrium in (5.3) will not be a simple superposition of normal modes.

The nature of the spectrum when streaming and collisions are coupled is more complex. For simplicity consider an initial value problem for a single wave number k,

$$\frac{\partial f}{\partial t} + i(\xi \cdot k)f + L(f) = 0 \ . \tag{5.5}$$

Now the relevant operator is $i(\xi \cdot k) + L$ and the continuum is the set of complex values taken by $i(\xi \cdot k) + \nu(\xi)$.[46] Also, the point eigenvalues of L at the origin are perturbed to neighboring points in the right half plane.[46] If the intermolecular potential is hard and k is small, the asymptotic behavior after a long time will be dominated by these point eigenvalues and will approximate the Navier-Stokes asymptotic behavior. If the potential is soft or if k is large, the continuum will dominate the ultimate behavior which will bear no resemblance to fluid dynamics. However, if k is small, even with the soft potential (continuum extending to the origin) there will be an <u>intermediate</u> time during which the point spectrum dominates (fluid-like behavior), although the ultimate

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46. This is a plausible mathematical conjecture but is not a consequence of known theorems.

complicated behavior will be dominated by the continuum.

The situation is not quite the same in a boundary value problem. For example, in a half space we distinguish time-independence

$$(\xi \cdot \nabla)f + L(f) = 0 \tag{5.6}$$

(e.g. heat flow or shear flow) from the case of a vibrating wall

$$i\omega f + (\xi \cdot \nabla)f + L(f) = 0 \tag{5.7}$$

(sound propagation at given ω). In these cases the continuum will consist of the values taken by $\nu(\xi)/\xi$ or $(\nu+i\omega)/\xi$ respectively.[46] In either case the continuum extends to the origin ($\nu/\xi \longrightarrow 0$ for large ξ) with the single exception of the rigid sphere molecule. In the steady problem (5.6), the degenerate point eigenvalue at the origin dominates far away from the wall and gives a linear profile of temperature and velocity. But the continuum which extends to the origin implies that the rate at which the solution approaches this linear profile is not a simple exponential. With the oscillating wall (5.7), the continuum always dominates sufficiently far from the wall, and the asymptotic behavior has no relation to the fluid (Navier-Stokes) solution of the same problem. More important, as a consequence of the continuum, the ultimate decay will not be a property of the gas alone (as it is for a

point eigenvalue) but will depend on details of the boundary condition at the wall. On the other hand, if k is small, there will be an intermediate region where the displaced point spectrum (only slightly damped) dominates and is approximately fluid dynamic.

These boundary value problems have been successfully solved for a model of the Boltzmann equation (for shear flow by Cercignani[14] and for sound dispersion by Weitzner[15]) and the qualitative features just described (which should remain true for the actual Boltzmann equation) have been found.

Perhaps the most significant feature of the appearance of a continuous spectrum is that there is no universal asymptotic decay (either in time or in space), but the rate of decay depends on the smoothness of the data (initial or boundary respectively[47]). Quantitative solution of the actual Boltzmann equation in boundary value problems is a difficult matter. But even this qualitative conclusion may turn out to be verifiable experimentally. In an experiment with an oscillating crystal, if ω is taken to be comparable to or higher than the mean collision frequency, one should observe an asymptotic decrement which is not reproducible if the preparation or composition of the crystal surface is altered. This would qualitatively contradict not only the Navier-Stokes solution

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47. In a neutral gas see Ref. 44; for the equivalent problem in a plasma ("Landau" damping), see Harold Weitzner, Phys. Fluids 6, 1123 (1963).

but any solution arising from a single ultimately dominant normal mode.

With regard to potentials of infinite total cross-section, only one result is known, viz., that there is a pure point spectrum for Maxwellian molecumes in the spatially homogeneous problem.[7] What the spectrum is in a spatially dependent problem for the Maxwellian potential is not certain, and the spectrum for any other infinite range potential even in the spatially homogeneous case is impossible to guess because the cutoff Maxwellian is borderline and separates the case of a continuum which extends to infinity from the case of a continuum which extends to the origin.

6. Conclusions

Recent developments in the theory of the Boltzmann equation indicate a trend away from *ad hoc* and heuristic approximation methods and towards more precise treatments which pay more attention to the mathematical idiosyncracies of individual problems. There are a number of directions in which one can hope to see further advances of this type within the near future.

First we mention the influence of a continuous spectrum which, although possibly even qualitatively observable

experimentally (cf. Sect.5) requires quantitative exploitation. Up to now, this has only been done with models of the Boltzmann equation which do not permit quantitative experimental comparisons.

A rigorous theory of the Boltzmann equation, which has thus far been successful only for cutoff potentials, should be extended to cases of infinite total cross-section. The principal purpose is mathematical, viz., to find out whether such a very singular form of the Boltzmann equation is a legitimate mathematical enterprise. For the same reason there is need for the development of strong nonlinear existence theorems even for cutoff potentials.

The asymptotic interpretation of the Hilbert and Chapman-Enskog theories requires completion to incorporate proper shock slip and boundary slip conditions. Although these problems are more difficult than the already solved initial slip problem, they should not be totally inaccessible.

At the opposite extreme, almost free flow, the accuracy of the first integral iteration suggests that extension of precise results from model equations to the actual Boltzmann equation will be quickly forthcoming.

Date Due DEC 7 1964

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AUTHOR
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JUL 17 1975	LAM	

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